ANALYTIC ITERATION AND DIFFERENTIAL EQUATIONS*

BY

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ABSTRACT

In this paper we study some mapping properties of analytic iterations $W(a, z)$. Our purpose is to establish a sufficient condition for $W(a, z)$ to be conformal and univalent in z for $z \in D$, where D is a given domain and for sufficiently small $|a|$. To this end we consider the differential equation $\partial W(a, z)/\partial a = L[W(a, z)]$ with the condition $W(O, z) = z$. A sufficient condition for the solution $W(a, z)$ of this system to be conformal and univalent in D for $|a| < a_0$ (for some $a_0 > 0$), and to satisfy the iteration equation, is established.

1. **Introduction and plan.** We are concerned with functions $W(a, z)$ analytic in a and z, which satisfy the iteration equation

$$W[a, W(b,z)] = W(a + b, z), \tag{1}$$

with

$$W(O, z) = z, \tag{2}$$

for $z \in D$, where D is a given domain, and for sufficiently small $|a|, |b|$ and $|a + b|$.

Putting

$$\left.\frac{\partial W(a,z)}{\partial a}\right|_{a=0} = L(z), \tag{3}$$

it is known [3], [2], that the function $W(a, z)$ satisfies simultaneously the following three differential equations:

$$\frac{\partial W(a,z)}{\partial a} = L[W(a,z)], \tag{4}$$

$$\frac{\partial W(a,z)}{\partial a} = \frac{\partial W(a,z)}{\partial z} L(z), \tag{5}$$

and hence also:

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$$\frac{\partial W(a,z)}{\partial z} = \frac{L[W(a,z)]}{L(z)}. \tag{6}$$

Evidently it follows from the definitions, that if the mappings $W(a,z)$ (for sufficiently small $|a|$) map D conformally on D_a, then $L(z)$ is necessary regular in D. Our main purpose is to establish a sufficient condition for the mappings $W(a,z)$ to be *conformal* in D for $|a| < a_0$, for some $a_0 > 0$. Next we ask about the sufficient condition for the mappings $W(a,z)$ to be *univalent* in D. It turns out that the answer to both questions is the same, namely: If $L(z)$ is regular and single-valued in the closure $\bar{D}$ of the domain D, with a double pole at most at $z = \infty$ (in the case $\infty \in \bar{D}$), then $W(a,z)$, for sufficiently small $|a|$, map D conformally and univalently onto D_a. Now, let $W(a,z)$ (for $z \in D$) be a single-valued analytic function of a for $a \in A$, where A is a bounded domain in the a plane including the origin. Moreover, let $W(a,z)$ satisfy equations (1) and (2) for $z \in D$ and $a, b, a+b \in A$. Consider now the mapping of D given by $W(a^*,z)$, where $a^* \in A$. If there exists a continuous curve $C \subset A$ connecting a^* with the origin and such that for every $a \in C$, $L(z)$ is regular and single-valued in $\bar{D}_a$ (with a double pole at most at $z = \infty$), then $W(a^*,z)$ maps D conformally and univalently onto D_{a^*}.

In the following we assume that it is the function $L(z)$, rather that $W(a,z)$, that is given, and we shall use the differential systems (4) and (2) or (5) and (2) to generate the function $W(a,z)$, which is obtained as the solution of either system. We shall prove that this solution, $W(a,z)$, satisfies equation (1) and is conformal and univalent in D for $|a| < a_0$ for some $a_0 > 0$.

We do not treat the differential equation (6), as this has been done (at least in the special case when $L(0) = L'(0) = 0$) in [2], but we obtain Theorem 1 of [2] as an immediate corollary of our Theorem 1.

We first suppose D to be bounded and later the results are extended, with some modifications, to the case of an unbounded domain D.

2. **The case of a bounded domain** D. We consider the differential equation (4) with the initial condition (2). Equation (4) can be regarded as an ordinary differential equation and we may apply the classical existence and uniqueness theorem [1]:

Let $L(W)$ be a regular and single-valued function in the neighborhood $|W - z| \leqq \rho(z)$ of the initial value z, then there exists one and only one analytic function $W(a,z)$ which is regular for $|a| < a_0(z)$ and such that:

(i) $W(0,z) = z$.

(ii) $|W(a,z) - z| \leqq \rho(z)$ for $|a| < a_0(z)$.

(iii) $\dfrac{\partial W(a,z)}{\partial a} = L(W)$ for $|a| < a_0(z)$.

Using Picard's successive approximation method, and the fact that the right hand side of (4) does not contain the independent variable a explicitly (a fact which will play an important role in the sequel) one obtains $a_0(z) = \rho(z)/M(z,\rho)$, where $M(z,\rho)$ is the maximum of $|L(W)|$ for $|W - z| \leqq \rho(z)$.

We now restrict the initial values z to a bounded domain D, such that the function $L(z)$ is regular and single-valued in the closure $\bar{D}$ of D. There exists then a minimal distance $d > 0$ from $\bar{D}$ to the nearest singularity of $L(z)$. Denote by $D(R)$ the domain consisting of points W, for which $|W - z| < R$, $z \in \bar{D}$ and $0 < R < d$, where R is chosen sufficiently small to ensure that $L(z)$ is still regular and *single-valued* in the closure of $D(R)$. (The existence of such $R > 0$, follows from the fact that $L(z)$ is single-valued in the compact domain $\bar{D}$). Denote by $M(D,R)$ the bound of $|L(W)|$ for $W \in \overline{D(R)}$. We have then:

$$|L(W)| \leqq M(D,R) \text{ for } |W - z| \leqq R \text{ and } z \in \bar{D}. \tag{7}$$

We now prove:

THEOREM 1. *Let $L(z)$ be a regular single-valued function in the closure of the bounded domain D. Then there exists one and only one analytic function $W(a,z)$, satisfying equation* (4) *with the initial condition* (2) *for $z \in D$. For $z \in D$ and $|a| < a_0 = R/M(D,R)$, this unique solution $W(a,z)$ is regular in a, and regular and univalent in z. It satisfies equation* (1) *for $|a|$, $|b|$ and $|a + b| < a_0$ and $z \in D$.*

Proof. It follows from the existence and uniqueness theorem that for every initial value $z \in \overline{D(R)}$ there exists a unique solution of the differential equation (4) with the initial condition (2), which is regular for $|a| < a_0(z) = \rho(z)/M(z,\rho)$. If we restrict the initial values to $\bar{D}$, then $W(a,z)$ is regular in a for $|a| < R/M(D,R) = a_0$ and for every $z \in \bar{D}$ and satisfies:

$$|W(a,z) - z| \leqq R, \quad |a| < a_0, \qquad z \in \bar{D}. \tag{8}$$

Note that we may apply the same argument to points $z \in \overline{D(R)}$ and using the fact that the minimal distance from $\overline{D(R)}$ to the nearest singularity of $L(z)$ is $d - R$, we obtain that $W(a,z)$ is then regular in a for $|a| < \alpha_0$, for some $\alpha_0 > 0$.

It also follows from the classical proof of the existence and uniqueness theorem, that $W(a,z)$ is analytic in the initial value z. We prove it here again in order to obtain something more, namely that $W(a,z)$ is analytic in z for $z \in D$ and $|a| < a_0$.

We expand $W(a,z)$ as a power series in a with coefficients which are functions of z and put:

$$W(a,z) = \sum_{n=0}^{\infty} \frac{P_n(z)}{n!} a^n, \tag{9}$$

where $P_n(z) = \left.\frac{\partial^n W}{\partial a^n}\right|_{a=0}$. Using (4) and (2) we obtain:

$$P_0(z) = z,\ P_1(z) = L(z), \tag{10}$$

and generally:

$$P_n(z) = \frac{\partial^n W}{\partial a^n}\bigg|_{a=0} = \left\{\frac{\partial}{\partial W}\left(\frac{\partial^{n-1} W}{\partial a^{n-1}}\right) \cdot \frac{\partial W}{\partial a}\right\}\bigg|_{a=0} = P'_{n-1}(z)L(z). \tag{11}$$

It follows from (11) that $P_n(z)$ is a polynomial in $L(z)$ and its derivatives up to order $(n-1)$. $P_n(z)$ is, thus, a regular single-valued function for $z \in \bar{D}$. The series in (9) is a series of regular functions, which converges uniformly for $z \in \bar{D}$ and $|a| < a_0$. Hence $W(a,z)$ is regular in a and z for $|a| < a_0$ and $z \in D$.

To prove that $W(a,z)$ satisfies (1), we use the uniqueness property of the solution of the system (4) and (2). Suppose that $|a| < \alpha_0$, $|b| < a_0$ and $|a+b| < a_0$ and put $a + b = c$. Consider the function $W(c,z)$ once as a function of z and c, and once as a function of z and a with b kept constant. In the second case we have:

$$\frac{\partial W(c,z)}{\partial a} = \frac{\partial W(c,z)}{\partial c} = L[W(c,z)].$$

Hence $W(c,z)$, qua function of a and z satisfies the differential system:

$$\frac{\partial W(c,z)}{\partial a} = L[Wc,z)], \tag{4'}$$

with:

$$W(c,z)\big|_{a=0} = W(b,z), \tag{2'}$$

where $W(b,z)$ is uniquely defined as $|b| < a_0$ and $z \in D$, and by (8) we have $W(b,z) \in \overline{D(R)}$. But the unique solution of the system (4') and (2') can be written as $W[a, W(b,z)]$, which is regular for $|a| < \alpha_0$, and so $W(c,z) = W[a, W(b,z)]$, which is (1). But as equation (1) holds for $z \in D$, $|a| < \alpha_0$, $|b| < a_0$, $|c| = |a+b| < a_0$, it holds whenever both sides can be defined, i.e. at least for $z \in D$, $|a|$, $|b|$, $|a+b| < a_0$.

In fact we have used equation (1) to obtain an analytic continuation of $W(a,z)$, where $z \in D_b$ and $|b| < a_0$, for such values of a for which $|a+b| < a_0$. As the function $W(a,z)$, so continued satisfies equation (1), it also satisfies equation (4).

The univalence of $W(a,z)$ for $z \in D$ and $|a| < a_0$ can be proved either by using equation (1), or as another consequence of the uniqueness of the solution. Suppose there exist in D two distinct points z_1 and z_2, and a value $|b^*| < a_0$, such that:

$$W(b^*, z_1) = W(b^*, z_2) = W^*. \tag{12}$$

Put $c = a + b^*$ and consider the two functions $W(c,z_1)$ and $W(c,z_2)$. Both functions, as functions of a, satisfy equation (4) with the initial condition: $W\big|_{a=0} = W^*$. By the uniqueness property it follows that $W(c,z_1) = W(c,z_2)$ holds for $|c - b^*| < \alpha_0$, for some $\alpha_0 > 0$. But both functions are regular for

$|c| < a_0$, $a_0 > 0$ and they coincide in the disk $|c - b^*| < \alpha_0$. Hence, by the monodromy theorem, they coincide everywhere, and in particular for $c = 0$. Thus, $W(0, z_1) = W(0, z_2)$ which by (2) implies $z_1 = z_2$.

REMARK. The estimate $a_0 = R/M(D,R)$ obtained by Picard's method is sharp in the sense that it cannot be improved by a multiplicative constant. Indeed, let $L(W) = W^{-1/n}$ $n = 1, 2, 3, \cdots$ and let D be a bounded domain contained in the half plane $\mathrm{Re}\{z\} > 1$, with $z = 1$ as a boundary point. By solving equation (4) with condition (2) we obtain:

$$W(a,z) = z\left\{1 + \frac{(n+1)az^{-(n+1)/n}}{n}\right\}^{n/(n+1)}, \text{ where } 1^{n/(n+1)} = 1.$$

$W(a,z)$ is regular for $|a| < n/(n+1)|z|^{(n+1)/n}$, so that $W(a,z)$ is regular for $z \in D$ and $|a| < n/(n+1)$. In this case the value $a_0 = n/(n+1)$ is the best possible. On the other hand, for every $0 < R < 1$ we have $R/M(D,R) = R(1-R)^{1/n}$. We easily get:

$$\mathrm{Max}\ \frac{R}{M(D,R)} = \frac{n}{n+1}\left(\frac{1}{n+1}\right)^{1/n} = \frac{n\mu}{n+1}$$

where $\mu = (1/(n+1))^{1/n} < 1$ for every n, but $\lim_{n\to\infty} \mu = 1$. Hence if we put $a_0 = CR/M(D,R)$, we cannot take $C > 1$.

We deduce now from Theorem 1 two corollaries regarding the solutions of the differential equations (5) and (6).

COROLLARY 1. *Consider the partial differential equation* (5) *with the boundary condition* (2) *and let* $L(z)$ *be a regular single-valued function in the closure of the bounded domain* D. *Then, there exists a unique analytic solution* $W(a,z)$ *of the system* (5) *and* (2) *and it is given by the power series* (9); *i.e. it is identical with the solution of the system* (4) *and* (2). *Hence all the results obtained in Theorem* 1 *for the solution of* (4) *and* (2) *are valid for the solution of* (5) *and* (2).

Proof. Assume there exists a solution of the system (5) and (2), given by a power series in a:

$$W(a,z) = \sum_{n=0}^{\infty} \frac{Q_n(z)}{n!} a^n. \tag{13}$$

Inserting (13) into (5) and using (2) we obtain:

$$Q_0(z) = z,\ Q_n(z) = Q'_{n-1}(z)L(z), \qquad n = 1, 2, \cdots \tag{14}$$

By comparing (14) with (10) and (11) we see that $Q_n(z) \equiv P_n(z)$, $(n = 0, 1, 2, \cdots)$.

COROLLARY 2. *Theorem* 1 *of* [2]. *Let*

$$L(z) = l_1 z^2 + l_2 z^3 + \cdots. \tag{15}$$

In Theorem 1 [2], the authors prove that if (15) converges for $|z|<r, r>0$, then the series defines a function $L(z)$ and permits the construction of a uniquely defined function $W(a,z)$, satisfying equation (1). This function $W(a,z)$ is then analytic in a and z for all finite complex a and for $|z| < R(a)$, $R(a) > 0$. The construction of the function $W(a,z)$ is carried out by proving the existence of a solution of the differential equation (6), which, in this case, has an inconvenient singularity at $z = 0$. We give another proof to this theorem: If $L(z)$ is regular for $|z| < r$, $r > 0$, it follows from (15) that for $|z| \leqq r_1 < r$ we have $|L(z)| \leqq K|z|^2$ for some K. Let D be the disk $|z| < R \leqq r_1/2$, then $M(D,R) \leqq (2R)^2K$. Note that here we use R both as the radius of the disk D and as the radius of extension of D in (7). It follows that

$$a_0 = \frac{R}{M(D,R)} \geqq \frac{R}{4R^2K} = \frac{1}{4RK}.$$

If $R \to 0$, then $R/M(D,R) \to \infty$. This implies that for every finite a, there exists $R(a) > 0$, such that $|a| < 1/4R(a)K \leqq a_0$ holds. Thus, we have proved that for every finite a, there exists a disk $|z| < R(a)$, $R(a) > 0$, such that $W(a,z)$ is regular (and even univalent) for $|z| < R(a)$.

3. **Extension to an unbounded domain** D.

Theorem 2. *Let D be an unbounded domain in the z plane, such that $\bar{D}$ contains the point at infinity, but not the whole z plane, and let $L(z)$ be analytic and single-valued for $z \in \bar{D}$, with a double pole at most at $z = \infty$. Then there exists a unique solution of equation* (4) *with the initial condition* (2) *for $z \in D$. This solution $W(a,z)$ is analytic in a for $|a| < \tilde{a}_0$, $\tilde{a}_0 > 0$ and $z \in D$, conformal and univalent in z for $z \in D$, $|a| < \tilde{a}_0$, and has at most one simple pole at some point $z_0 \in D$, (z_0 may vary with a), and it satisfies equation* (1) *for $|a|$, $|b|$ and $(a+b) < \tilde{a}_0$.*

Proof. Without loss of generality we may assume $O \notin \bar{D}$. Setting

$$\omega = \frac{1}{W}, \xi = \frac{1}{z} \tag{16}$$

the system (4) and (2) transforms into:

$$\frac{d\omega}{da} = -\omega^2 L(\omega^{-1}) = L^*(\omega), \tag{17}$$

$$\omega(O) = \xi, \tag{18}$$

and the domain D is mapped by $\xi = 1/z$ onto the bounded domain Δ. the function $L^*(\omega)$ is regular in $\bar{\Delta}$, and we can apply Theorem 1 to the system (17) and (18). Thus, there exists a unique analytic solution $\omega(a,\xi)$ which is regular in a, regular and univalent in ξ for $|a| < \tilde{a}_0$, $\tilde{a}_0 > 0$, and $\xi \in \Delta$. By (16) this implies that $W(a,z)$, which satisfies (4) and (2), is regular in a, regular and univalent in z for $z \in D$ and $|a| < \tilde{a}_0$, unless $\omega(a,\xi) = 0$. But, since $\omega(a,\xi)$ is univalent in Δ (for $|a| < \tilde{a}_0$) it has at most one simple zero $\xi_0 \in \Delta$. (For every a the point ξ_0 may be different). Hence $W(a,z)$ has at most one simple pole at some point $z_0 = 1/\xi_0$ which belongs to D.

Note that if $L(z)$ is regular or has at most one simple pole at $z = \infty$, then $L^*(0) = 0$. Hence $\xi = 0$ is a fixpoint of $\omega(a,\xi)$, i.e. $\omega(a,0) = 0$, and by (16) it follows that $W(a,z)$ has one simple pole at $z = \infty$, so that $z = \infty$ is a fixpoint of $W(a,z)$.

By Theorem 1, $\omega(a,\xi)$ satisfies equation (1) under the specified conditions, and it easily follows that the same is true for $W(a,z)$.

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